Sec 2.1 Population models

$P(t) =$ popul.

$$\frac{dP}{dt} = [\beta - \delta]\, P(t) \qquad \frac{\text{births}}{\text{(pop)(time)}} \qquad \frac{\text{deaths}}{\text{(pop)(time)}}$$

Empirically (resource limitations?), populations grow with $\beta = \beta_0 - \beta_1 P(t)$.

$\delta = \delta_0$ (const.)

So a realistic model is $\frac{dP}{dt} = \beta_0 P - \beta_1 P^2 - \delta_0 P$

$$\Rightarrow \boxed{\frac{dP}{dt} = \underbrace{aP}_{\text{total birth rate}} - \underbrace{bP^2}_{\text{total death rate}}}$$

"the logistic eq"

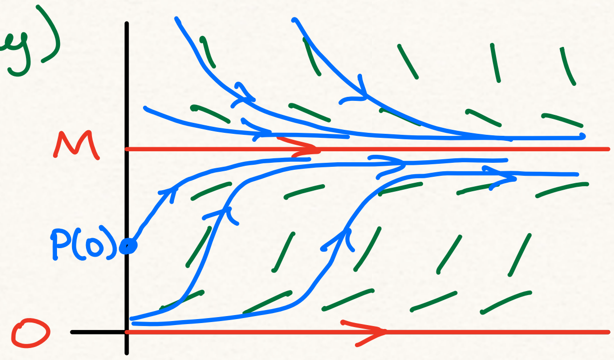
A very common alt. form: $\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$

$$\boxed{\frac{dP}{dt} = kP(M - P)}$$

"carrying capacity" or "limiting pop."

Observe

- $P' = 0$ if $P = 0$ or $P = M$
- $0 < P < M \Rightarrow P' > 0$ (growth)
- $P > M \Rightarrow P' < 0$ (decay)



Ex: a population initially ($t=0$) has

- 8 births/month
- 6 deaths/month
- init. pop. $P(0) = 120$

assuming population grows according to Logistic eq, what is the limiting pop.? How long until P reaches 95% of this limit?

Logistic Eq $\quad \frac{dP}{dt} = \underbrace{aP}_{\text{birth rate}} - \underbrace{bP^2}_{\text{death rate}} \quad$ (alt. $\frac{dP}{dt} = kP(M-P)$)

need a, b.

@ $t=0$, $\quad 8 = aP(0) = a \cdot 120 \Rightarrow \quad a = \frac{8}{120}$

" " $\quad 6 = bP(0)^2 = b \cdot (120)^2 \Rightarrow b = \frac{6}{(120)^2}$

$$\frac{dP}{dt} = aP - bP^2 = bP\left(\frac{a}{b} - P\right)$$
$$= kP(M-P) \qquad \boxed{k = b = \frac{6}{(120)^2}}$$

$$M = \frac{a}{b} = \frac{8}{120} \cdot \frac{(120)^2}{6} = \quad 120 \cdot \frac{8}{6} = \boxed{160 = M}$$

So ODE is
$$\frac{dP}{dt} = \frac{6}{(120)^2} P(160 - P)$$

Need solution for logistic eq $\frac{dP}{dt} = kP(160-P)$

$$\int \frac{dP}{P(160-P)} = \int k\,dt$$

separable
(also Bernoulli)

Need partial fraction: $\frac{1}{P(160-P)} = \frac{A}{P} + \frac{B}{160-P}$

$$0P + 1 = A(160-P) + BP = \underbrace{160A}_{=1} + \underbrace{(B-A)}_{=0}P$$

So $A = \frac{1}{160}$, $B = A = 1/160$.

$$\int \frac{dP}{P(160-P)} = kt + C$$

$$\frac{1}{160}\int \frac{dP}{P} + \frac{1}{160}\int \frac{dP}{160-P} = kt + C$$

$$\frac{1}{160}\left(\ln|P| - \ln|160-P|\right) = kt + C$$

$$\ln\left|\frac{P}{160-P}\right| = 160kt + C$$

$$\Longrightarrow \frac{P}{160-P} = Ce^{160kt}$$

$k = \frac{6}{(120)^2}$

$160k = \frac{16\cdot 6\cdot 1\cancel{0}}{12\cdot 12\cdot \cancel{10}\cdot 10}$

$= \frac{1}{2}\cdot\frac{1}{10}\cdot\frac{4}{3}$

$= \frac{4}{60} = \frac{1}{15}$

(impl. sol.) $\longrightarrow$ $\Longrightarrow \frac{P}{160-P} = Ce^{t/15}$

Need $P(0) = 120 \longrightarrow (t=0,\ P=120)$

$$\frac{120}{40} = Ce^{0} \longrightarrow C = 3$$

$$\frac{P}{160-P} = 3e^{t/15}$$

$$\ldots \quad P = \frac{480e^{t/15}}{1+3e^{t/15}} \cdot \frac{e^{-t/15}}{e^{-t/15}}$$

$$\boxed{P = \frac{480}{3+e^{-t/15}}}$$

Lastly want $P = (0.95)(160) = \dfrac{480}{3+e^{-t/15}}$

$\ldots$ $\underline{t \approx 28 \text{ months}}$